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Homotopy Methods for Solving Variational Inequalities in Unbounded Sets

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Abstract. In this paper, for solving the finite-dimensional variational inequality problem

$$(x - x^*)^T F(x^*) \geqslant 0, \quad \forall x \in X,$$

where F is a C^r $(r > 1)$ mapping from X to R^n, $X = \{x \in R^n : g(x) \leqslant 0\}$ is nonempty (not necessarily bounded) and $g(x) : R^n \to R^m$ is a convex C^{r+1} mapping, a homotopy method is presented. Under various conditions, existence and convergence of a smooth homotopy path from almost any interior initial point in X to a solution of the variational inequality problem is proven. It leads to an implementable and globally convergent algorithm and gives a new and constructive proof of existence of solution.

Key words: Homotopy method, Interior point method, Variational inequality

1. Introduction

Solving a finite-dimensional variational inequality is to find a vector $x^* \in X \subset R^n$ such that

$$(x - x^*)^T F(x^*) \geqslant 0, \quad \forall x \in X, \tag{1}$$

where X is a nonempty, closed and convex subset of R^n and F is a mapping from R^n to itself, denoted by $\mathrm{VI}(X, F)$.

The variational inequality problem (VIP) has had many successful practical applications in the last three decades (see, e.g. [1–4]). It has been used to formulate and investigate equilibrium models arising in economics, transportation, regional science and operations research. So far, a large number of existence conditions have been developed in the literature (e.g. [5–10]). Harker and Pang [11, 12] gave excellent surveys of theories, methods and applications of VIPs.

The history of algorithms for solving the finite-dimensional variational inequality is relatively short. Major algorithms such as Newton's method are locally convergent. However, generally, it is difficult to know a good initial point hence locally convergent algorithms can not be applied. Thus it is necessary to construct globally convergent algorithms. Only for $F(x)$ with very special properties some globally convergent algorithms have been given.

Homotopy method (see, [13, 14] for introductions) which has paid much attention since 1970's is a class of important globally convergent method. Many homotopy method have been given to constructively prove existence of solution and to serve as implementable algorithms for nonlinear systems, fixed point problems, nonlinear programming and complementarity problems.

Recently, utilizing the combined homotopy (see [15, 16]), Lin and Li [17] gave a homotopy method for VIP in a bounded set X. It was also conjectured in [17] that the result can be generalized to a VIP in an unbounded set, however, up till now, no such result has been given.

In this paper, we will discuss about homotopy methods for VIPs in an unbounded set. Under conditions which are commonly used in the literature, a smooth path from a given interior point of X to a solution of VIP will be proven to exist. This will give constructive proof of existence of solution and lead to an implementable globally convergent algorithm to the VIP.

In Section 2, we formulate an equivalent form of VIP (K-K-T condition) and list some lemmas from differential topology which will be used in this paper. In Section 3, we give the homotopy and prove in detail existence of the smooth path from a given point in X to a solution of the VIP under a weak condition. Then we give some corollaries, with only key points of proof, to show that similar results can obtained for VIPs under many other commonly used conditions.

2. Preliminary Lemmas

In this paper, we restrict the feasible set X to as follows:

$$X = \{x \in R^n : g(x) \leqslant 0\}, \tag{2}$$

where $g(x) = (g_1(x), \ldots, g_m(x))^T$ and g_i's are assumed to be convex.

Let X^0 be the strictly feasible set of (1), i.e.,

$$X^0 = \{x \in R^n : g(x) < 0, i = 1, \ldots, m\}.$$

We assume that the Slater constraint qualification holds for X, i.e., there exists a point $x^0 \in X$ such that $g(x^0) < 0$.

Let R^m_+ and R^m_{++} denote the nonnegative and positive orthant of R^m, $\partial X = X - X^0$. Let

$$I(x) = \{i \in \{1, \ldots, m\} : g_i(x) = 0\} \tag{3}$$

be the active index set at $x \in X$.

In [17], a homotopy method for $\mathrm{VI}(X, F)$ with bounded X was given. In this paper, we will discuss $\mathrm{VI}(X, F)$ with X which is not necessarily bounded.

The following lemma formulates a equivalent form of VI(X,F).

LEMMA 2.1 (See [11]). *Let F be a continuous mapping from R^n to itself, X be defined by (2) and functions $g_i(x), i = 1, \ldots, m$ are twice continuously differentiable and convex. Then $x^* \in X$ is a solution to $VI(X, F)$ if, and only if, there exists a vectors $\lambda^* \in R^m_+$ such that*

$$\begin{aligned} &F(x^*) + \nabla g(x^*)\lambda^* = 0, \\ &\lambda_i^* g_i(x^*) = 0, \; i = 1, \ldots, m. \end{aligned} \tag{4}$$

The following lemmas from differential topology will be used in the next section. At first, let $U \subset R^n$ be an open set, let $\phi : U \to R^p$ be a C^α ($\alpha > \max\{0, n-p\}$) mapping, we say that $y \in R^p$ is a regular value for ϕ, if

$$\text{Rang}[\partial\phi(x)/\partial x] = R^p, \quad \forall x \in \phi^{-1}(y).$$

LEMMA 2.2 (See [13]). *Let $V \subset R^n$, $U \subset R^m$ be open sets, and let $\phi : V \times U \to R^k$ be a C^α mapping, where $\alpha > \max\{0, m-k\}$. If $0 \in R^k$ is a regular value of ϕ, then for almost all $a \in V$, 0 is a regular value of $\phi_a = F(a, \cdot)$.*

LEMMA 2.3 (See [18]). *Let $\phi : U \in R^n \to R^p$ be a C^α ($\alpha > \max\{0, n-p\}$) mapping. If 0 is a regular value of ϕ, then $\phi^{-1}(0)$ consists of some $(n-p)$-dimensional C^α manifolds.*

LEMMA 2.4 (See [18]). *A one-dimensional smooth manifold is diffeomorphic to a unit circle or a unit interval.*

3. Main Results

THEOREM 3.1. *Suppose that*

(A) *$g_i(x)$, $i = 1, \ldots, m$ are convex C^{r+1} ($r > 1$) functions and X^0 is nonempty.*

(B) *$\forall x \in \partial X$, $\{\nabla g_i(x) : i \in I^0(x)\}$ is linear independent.*

Let $F : X \to R^m$ be a C^r mapping satisfying the following condition

(C) *There exists some $z^0 \in X$ such that the set*

$$X(z^0) = \{x \in X : (x - z^0)^T F(x) < 0\} \tag{5}$$

is bounded, i.e., there exists an $M > 0$, such that $\forall x \in X(z^0)$, $\|x\| \leqslant M$.

We have the following results:

(1) *There exists an $x^* \in X$, such that*

$$(x - x^*)^T F(x^*) \geqslant 0, \quad \forall x \in X,$$

(2) *For almost all $x^0 \in X^0$, $y^0 \in R^m_{++}$, the homotopy equation*

$$H(w^0, w, \mu) = \begin{pmatrix} (1-\mu)(F(x) + \nabla g(x)y) + \mu(x - x^0) \\ Yg(x) - \mu Y^0 g(x^0) \end{pmatrix} = 0, \tag{6}$$

where $w = (x, y)$, $w^0 = (x^0, y^0)$, $g(x) = (g_1(x), \ldots, g_m(x))^T$, $y = (y_1, \ldots, y_m)^T$, $Y = \text{diag}(y)$, $\nabla g(x) = (\nabla g_1(x), \ldots, \nabla g_m(x))$, *determines a smooth* *curve* $\Gamma_{w^0} \subset X^0 \times R^m_{++} \times (0,1]$ *starting from* $(w^0, 1)$. *As* $\mu \to 0$, *the limit set* $T \subset X \times R^m_+ \times \{0\}$ *of* Γ_{w^0} *is nonempty and the x-component of any point in* T *is a solution of the* $VI(X, F)$.

First of all, we prove the following three lemmas. For a given $w^0 \in X^0 \times R^m_{++}$, rewrite $H(w^0, w, \mu)$ in (6) as $H_{w^0}(w, \mu)$. Set

$$H^{-1}_{w^0}(0) = \{(w, \mu) \in X^0 \times R^m_{++} \times (0,1] : H_{w^0}(w, \mu) = 0\} \tag{7}$$

LEMMA 3.1. *If the conditions* (A) *and* (B) *of Theorem* 3.1 *hold, then for almost all* $w^0 \in X^0 \times R^m_{++}$, 0 *is a regular value of* $H_{w^0} : X^0 \times R^m_{++} \times (0,1] \to R^{m+n}$ *and* $H^{-1}_{w^0}(0)$ *consists of some smooth curves. Among them, a smooth curve* Γ_{w^0} *starts from* $(w^0, 1)$.

Proof. $\forall w^0 \in X^0 \times R^m_{++}$ and $\mu \in (0,1]$

$$\partial H(w^0, w, \mu)/\partial w^0 = \begin{bmatrix} -\mu I & 0 \\ -\mu Y^0 \nabla g(x^0)^T & -\mu G(x^0) \end{bmatrix},$$

where I is the identical matrix and $G(x^0) = \mathrm{diag}(g(x^0))$. By a simple computation:

$$|\partial H(w^0, w, \mu)/\partial w^0| = (-1)^{n+m} \mu^{m+n} \prod_{i=1}^{m} g_i(x^0).$$

From $x^0 \in X^0$, we have $g_i(x) < 0$, and hence

$$|\partial H(w^0, w, \mu)/\partial w^0| \neq 0.$$

Thus, 0 is a regular value of $H(w^0, w, \mu)$. By Lemma 2.2 and Lemma 2.3, for almost all $w^0 \in X^0 \times R^m_{++}$, 0 is a regular value of $H_{w^0}(w, \mu)$ and $H^{-1}_{w^0}(0)$ consists of some smooth curves. And, because

$$H_{w^0}(w^0, 1) = 0,$$

there must be a smooth curve Γ_{w^0} in $H^{-1}_{w^0}(0)$ starting from $(w^0, 1)$. □

LEMMA 3.2. *Suppose that the conditions of Theorem 3.1 hold. For a given* $w^0 \in X^0 \times R^m_{++}$, *if 0 is a regular value of* H_{w^0}, *then the projection of the smooth curve* $\Gamma_{w^0} \subset H^{-1}_{w^0}(0) = \{(w, \mu) \in X^0 \times R^m_{++} \times (0,1] : H_{w^0}(w, \mu) = 0\}$ *on the x-plane is bounded.*

Proof. If there exists a sequence $(x^k, y^k, \mu_k) \in \Gamma_{w^0}$, such that $\|x^k\| \to \infty$. From the first equality of (6), we have:

$$(1 - \mu_k)(F(x^k) + \nabla g(x^k) y^k) + \mu_k (x^k - x^0) = 0. \tag{8}$$

Since $g(x)$ is convex, the following inequalities hold:

$$g(x^0)^T \geqslant g(x)^T + (x^0 - x)^T \nabla g(x), \quad \forall x \in X,$$

i.e.,

$$(x^0 - x)^T \nabla g(x) \leqslant g(x^0)^T - g(x)^T. \tag{9}$$

By (8) and (9), we have

$$(1 - \mu_k)((x^k - z^0)^T F(x^k) + (x^k - z^0)^T \nabla g(x^k) y^k) + \mu_k (x^k - z^0)^T (x^k - x^0) = 0,$$

hence,

$$\begin{aligned}
&(1-\mu_k)(x^k - z^0)^T F(x^k)\\
&= -\mu_k(x^k - z^0)^T(x^k - x^0) - (1-\mu_k)(x^k - z^0)^T \nabla g(x^k) y^k\\
&= -\mu_k\|x^k - x^0\|^2 + \mu_k(x^k - x^0)^T(z^0 - x^0) + (1-\mu_k)(z^0 - x^k)\nabla g(x^k)y^k\\
&\leqslant -\frac{1}{2}\mu_k(\|x^k - x^0\|^2 - \|z^0 - x^0\|^2) + (1-\mu_k)(g(z^0) - g(x^k))^T y^k.
\end{aligned}$$

From the second equality of (6) and $g(z^0) \leqslant 0$, $y^k > 0$, we have

$$(1-\mu_k)(x^k - z^0)^T F(x^k) \leqslant -\frac{1}{2}\mu_k(\|x^k - x^0\|^2 - \|z^0 - x^0\|^2 - (1-\mu_k)g(x^0)^T y^0). \tag{10}$$

If $\|x^k\| \to \infty$, because $\|z^0 - x^0\|^2$, $g(x^0)$ and y^0 are constant and $1-\mu_k$ is bounded, there exists k such that $\|x^k\| > M, \mu_k \in (0,1]$, and the right-hand side of (10) is strictly smaller than 0, i.e.,

$$(x^k - z^0)^T F(x^k) < 0.$$

This contradicts with the condition (C). So $\{x^k\}$ is bounded. □

LEMMA 3.3. *Suppose that the conditions of Theorem* 3.1 *hold. For a given* $w^0 \in X^0 \times R^m_{++}$, *if* 0 *is a regular value of* H_{w^0}, *then* Γ_{w^0} *is a bounded curve in* $X^0 \times R^m_{++} \times (0,1]$.

Proof. If $\Gamma_{w^0} \subset X^0 \times R^m_{++} \times (0,1]$ is an unbounded curve, then because we have proven in Lemma 3.2 that the projection of the smooth curve Γ_{w^0} on the x-plane is bounded, there exists a sequence of points $\{(x^k, y^k, \mu_k)\} \subset \Gamma_{w^0}$ and a nonempty index set $I^* \subset \{1, \ldots, m\}$, such that $x^k \to x^*$, $\mu_k \to \mu_*$, $y_i^k \to y_i^*$ for $i \notin I^*$ and $y_i^k \to +\infty$ for $i \in I^*$.

From the second equality of (6)

$$Y^k g(x^k) = \mu_k Y^0 g(x^0), \tag{11}$$

we have

$$I^* \subset I(x^*).$$

(i) When $\mu^* = 1$, rewrite (8) as

$$\sum_{i \in I(x^*)} (1-\mu_k)\nabla g_i(x^k) y_i^k + x^k - x^0 = (1-\mu_k)\left[-\sum_{i \notin I(x^*)} y_i^k \nabla g_i(x^k) - F(x^k) + x^k - x^*\right] \tag{12}$$

Since $\{x^k\}$ and $\{y_i^k\}$, $i \notin I(x^*)$ are bounded, as $k \to \infty$, (12) becomes

$$\lim\left[\sum_{i \in I(x^*)} (1-\mu_k) y_i^k \nabla g_i(x^k) + x^k - x^0\right] = 0.$$

Using $x^k \to x^*$, as $k \to \infty$, we have

$$x^0 = x^* + \sum_{i\in I(x^*)} \lim\left[(1-\mu_k)y_i^k\right]\nabla g_i(x^*). \tag{13}$$

Because I^* and hence $I(x^*)$ is nonempty, we know that $x^* \in \partial X$. Since $(1-\mu_k)y^k \geqslant 0$,

$$x^* + \sum_{i\in I(x^*)} \left[\lim(1-\mu_k)y_i^k\right]\nabla g_i(x^*)$$

is in the translated normal cone of X at x^*. Because x^0 is an interior point and X is convex, (13) is impossible.

(ii) When $\mu^* < 1$, rewrite (9) as

$$(1-\mu_k)(F(x^k)+\sum_{i\notin I^*}\nabla g_i(x^k)y_i^k)+\mu_k(x^k-x^0)+(1-\mu_k)\sum_{i\in I^*}\nabla g_i(x^k)y_i^k=0. \tag{14}$$

From $y_i^k \to +\infty$ for $i \in I^*$ and condition (B), it follows that the third part in left hand side of (14) tends to infinity, while the first and second parts are bounded. This is also impossible. Thus, Γ_{w^0} is bounded. □

Proof of Theorem 3.1. By Lemma 2.4, Γ_{w^0} must be diffeomorphic to a unit circle or a unit interval (0,1].

Since the matrix

$$\partial H(w^0, w, \mu)/\partial w^0 = -\begin{pmatrix} I & 0 \\ Y^0\nabla g(x^0)^T & G(x^0) \end{pmatrix}$$

is nonsingular, Γ_{w^0} is diffeomorphic to (0,1]. As $\mu \to 0$, the limit points of Γ_{w^0} belong to $\partial(X \times R_+^m \times (0,1])$. Let (w^*, μ^*) be a limit points of Γ_{w^0}, then only the following four cases are possible:

(i) $(w^*, \mu^*) \in X^0 \times R_{++}^m \times \{1\}$
(ii) $(w^*, \mu^*) \in \partial(X \times R_{++}^m) \times \{1\}$
(iii) $(w^*, \mu^*) \in \partial(X \times R_{++}^m) \times (0,1)$
(iv) $(w^*, \mu^*) \in X \times R_{++}^m \times \{0\}$

Since the equation $H_{w^0}(w^0, 1) = 0$ has only one solution $(w^0, 1)$ in $X^0 \times R_{++}^m \times \{1\}$, case (i) is impossible.

In cases (ii) and (iii), there must exist a sequence of $\{(x^k, y^k, \mu_k)\} \subset \Gamma_{w^0}$ such that $\|x^k\| \to \infty$ or, $g_i(x^k) \to 0$ for some $1 \leqslant i \leqslant m$ and $\|y^k\| \to \infty$. This contradicts with Lemma 3.2 or 3.3.

As a conclusion, case (iv) is the only possible case, and hence (x^*, y^*) is a solution of (4). By Lemma 2.1, x^* is a solution of the VI(X, F). □

REMARK 3.1. If X is a bounded set, the condition (C) of Theorem 3.1 holds obviously. Hence, the result of Theorem 3.1 implies the one in [17].

DEFINITION 3.1. A mapping $F: R^n \to R^n$ is said to be uniform diagonally dominant with respect to X if, for any distinct x, y in X and index i with $|x_i - y_i| = \|x - y\|_\infty$, there exists a positive scalar c such that

$$(x_i - y_i)(F_i(x) - F_i(y)) \geqslant c\|x - y\|_\infty^2, \tag{15}$$

where $\|\cdot\|_\infty$ denote the max norm.

DEFINITION 3.2. The mapping $F: R^n \to R^n$ is said to be

(a) pseudo-monotone over X if

$$F(y)^T(x - y) \geqslant 0 \text{ implies } F(x)^T(x - y) \geqslant 0, \quad \forall x, y \in X,$$

(b) uniform P-function on X if there exists a scalar $\alpha > 0$ such that

$$\max_{1 \leqslant i \leqslant n} (F_i(x) - F_i(y))^T(x - y) \geqslant \alpha\|x - y\|_2^2, \quad \forall x, y \in X,\ x \neq y, \tag{16}$$

(c) coercive with respect to X if there exists a vector $x^0 \in X$ such that

$$\lim_{x \in X, \|x\| \to \infty} \frac{(F(x) - F(x^0))^T(x - x^0)}{\|x\|} = +\infty, \tag{17}$$

where $\|\cdot\|$ denotes any vector norm in R^n.

(d) strongly monotone over X if there exists an $\alpha > 0$ such that

$$(F(x) - F(y))^T(x - y) \geqslant \alpha\|x - y\|^2, \quad \forall x,\ y \in X. \tag{18}$$

DEFINITION 3.3.

(a) A mapping $F: R^n \to R^n$ is said to be proper at the point $x^0 \in X$ if the set $L(x^0, X) = \{x \in X : (x - x^0)^T F(x^0) \leqslant 0\}$ is bounded.

(b) A mapping $F: R^n \to R^n$ is said to be weakly proper at the point $x^0 \in X$ if, for each sequence $\{x^k\} \subset X$ with the property $\|x^k\| \to \infty$ as $k \to \infty$, there exists some k such that

$$F(x^0)^T(x^k - x^0) \geqslant 0 \text{ and } \|x^k\| > \|x^0\|.$$

LEMMA 3.4 (See [19]). *Let $g: R^n \to R^m$ be defined as*

$$g(x) = (g_1(x_{k_1}), \ldots, g_m(x_{k_m}))^T. \tag{19}$$

Then, for any $\lambda \in R^m$, $z \in R^n$ and $x \in R^n$, we have

$$z_p(\nabla g(x)\lambda)_p = \begin{cases} 0, & \text{if } p \neq k_i,\ (i = 1, \ldots, m) \\ \lambda_i(\nabla g(x)z)_i, & \text{if } p = k_i, \end{cases}$$

for $p = 1, \ldots, n$.

COROLLARY 3.1. *Suppose that the conditions* (A) *and* (B) *of Theorem* 3.1 *hold. Let $F: X \to R^n$ be a C^r mapping, X be a rectangular set in R^n and one of the following conditions holds*:

(a) *F is a uniform diagonally dominant function with respect to X.*
(b) *F is a uniform P-function with respect to X and* $0 \in X$.

Then the conclusion of the Theorem 3.1 *holds.*

Proof. Since each rectangular set can be represented in the form of (2) with $g(x)$ in the form of (19), without loss of generality we assume that X is given by (2) and (19).

(a) If we can prove that the projection of the smooth curve $\Gamma_{w^0} \subset H_{w^0}^{-1}(0)$ on the x-plane is bounded under the supposed conditions, then the desired result can be shown by the similar argument as the proof of Theorem 3.1.

Suppose that there exists a sequence of x^k in the projection of the smooth curve Γ_{w^0} on the x-plane, such that $\|x^k\| \to \infty$.

By (9) and the second equality of (6), we have

$$\begin{aligned}(y^k)_i[\nabla g(x^k)^T(x^k - x^0)]_i &\geqslant (y^k)_i(g_i(x^k) - g_i(x^0)) \\ &\geqslant \mu_k(y^0)_i g_i(x^0),\end{aligned} \tag{20}$$

for all $i = 1, \ldots, m$.

By (20) and Lemma 3.4, we deduce that

$$(x^k - y)_i[\nabla g(x^k)y^k]_i \geqslant \mu_k(y^0)_i g_i(x^0). \tag{21}$$

Since $\{x^k\}$ is an infinite sequence, there exists a subsequence $\{x^{k_i}\}$ and some fixed index $l \in \{1, \ldots, m\}$, such that,

$$|x_l^{k_i} - y_l| = \|x^{k_i} - y\|_\infty, \quad \forall k_i.$$

Noticing that F is a uniform diagonally dominant function, we have

$$(x^{k_i} - y)^T(F_l(x^{k_i}) - F_l(y)) \geqslant c\|x^{k_i} - y\|_\infty^2, \quad \forall k_i. \tag{22}$$

By using the first equality of (6) and (21), we have

$$\begin{aligned}&(1-\mu_{k_i})[F_l(x^{k_i}) - F_l(x^0)](x^{k_i} - x^0)_l \\ &= (1-\mu_{k_i})F_l(x^{k_i})(x^{k_i} - x^0)_l - (1-\mu_{k_i})F_l(x^0)(x^{k_i} - x^0)_l \\ &= -(1-\mu_{k_i})[\nabla g(x^{k_i})^T y^{k_i}]_l(x^{k_i} - x^0)_l - \mu_{k_i}(x^{k_i} - x^0)_l^2 - (1-\mu_{k_i})F_l(x^0)(x^{k_i} - x^0)_l \\ &\leqslant -\mu_{k_i}(1-\mu_{k_i})(y^0)_l g_l(x^0) - \mu_{k_i}(x^{k_i} - x^0)_l^2 - (1-\mu_{k_i})F_l(x^0)(x^{k_i} - x^0)_l.\end{aligned}$$

Combining (22) and the above inequality yields

$$\begin{aligned}&(1-\mu_{k_i})c\|x^{k_i} - x^0\|_\infty^2 \\ &\leqslant -(1-\mu_{k_i})\mu_{k_i}(y^0)_l g_l(x^0) - \mu_{k_i}(x^{k_i} - x^0)_l^2 - (1-\mu_{k_i})F_l(x^0)(x^{k_i} - x^0)_l.\end{aligned}$$

When $\mu_{k_i} = 1$, we have

$$(x^{k_i} - x^0)_l^2 \leqslant 0,$$

since $\|x^{k_i}\| \to \infty$, this is impossible.

When $0 \leqslant \mu_{k_i} < 1$, we have

$$c \leqslant [-\mu_{k_i}(y^0)_l g_l(x^0) - F_l(x^0)(x^{k_i} - x^0)_l - \frac{\mu_{k_i}}{1-\mu_{k_i}}(x^{k_i} - x^0)_l^2]/\|x^{k_i} - x^0\|_\infty^2.$$

Since $\|x^{k_i}\| \to \infty$, the above inequality can not hold, we obtain a contradiction. Thus, we obtain that the projection of the smooth curve Γ_{w^0} on the x-plane is bounded. Hence, following the similar argument as the proof of Theorem 3.1, we can obtain the desired result.

(b) Suppose that there exists a sequence $\{x^{k_i}\}$ in the projection of the smooth curve Γ_{w^0} on the x-plane, such that $\|x^{k_i}\| \to \infty$. Let $y = 0$, there is a subsequence $\{x^{k_i}\}$ and some fixed index l, such that

$$[F_l(x^{k_i}) - F_l(0)]x_l^{k_i} = \max_{1 \leqslant i \leqslant n} [F_i(x^{k_i}) - F_l(x^0)]x_i^{k_i}, \quad \forall k_i.$$

Since $F(x)$ is a uniform P-function, we have

$$[F_l(x^{k_i}) - F_l(0)]x_l^{k_i} \geqslant c\|x^{k_i}\|^2.$$

Then, the rest part of the proof is the same as in (a). □

COROLLARY 3.2. *Assume that the conditions* (A) *and* (B) *of Theorem* 3.1 *hold. F is a C^r mapping and is coercive with respect to X, then the conclusion of Theorem 3.1 holds.*

Proof. Let $x^0 \in X$ satisfy (17). Suppose that there exists a sequence $\{x^k\}$ in the projection of the smooth curve Γ_{w^0} on the x-plane, such that $\|x^k\| \to \infty$. By the first equality of (6), we have

$$(1 - \mu_k)[(x^k - x^0)^T F(x^k) + (x^k - x^0)^T \nabla g(x^k) y^k] + \mu_k \|x^k - x^0\|^2 = 0. \tag{23}$$

For k large enough, $\mu_k \neq 1$, otherwise, by (23), $\|x^k - x^0\|^2 = 0$. This is impossible, so we have

$$(x^k - x^0)^T F(x^k) + (x^k - x^0)^T \nabla g(x^k) y^k \leqslant 0. \tag{24}$$

By (24), (9) and the second equality of (6), noticing that $g(x^0) \leqslant 0$, $y^0 \geqslant 0$ and $y^k \geqslant 0$, we have

$$\begin{aligned}
&(x^k - x^0)^T F(x^k) \\
&\leqslant (x^0 - x^k)^T \nabla g(x^k) y^k \\
&\leqslant [g(x^0) - g(x^k)]^T y^k \\
&\leqslant -\mu_k g(x^0)^T y^0.
\end{aligned}$$

Then, we have

$$\begin{aligned}
&(x^k - x^0)^T (F(x^k) - F(x^0))/\|x^k - x^0\| \\
&\leqslant -(\mu_k g(x^0)^T y^0 + (x^k - x^0)^T F(x^0))/\|x^k - x^0\|.
\end{aligned} \tag{25}$$

Since $g(x^0)$ and $F(x^0)$ are fixed, $\mu_k \in (0, 1]$ and $\|x^k\| \to \infty$ (as $k \to \infty$), the right hand side of (25) is bounded and left hand side of it goes to $+\infty$ from the coercivity of $F(x)$, which is impossible. Thus, the projection of the smooth curve Γ_{w^0} on the x-plane is bounded. The rest part of the proof is similar as that of Theorem 3.1. □

COROLLARY 3.3. *Assume that the conditions* (A) *and* (B) *of Theorem* 3.1 *hold. F is a C^r mapping that is strongly monotone over X. Then the conclusion of Theorem* 3.1 *holds.*

Proof. If F is strongly monotone over X, then F is coercive with respect to X. By corollary 3.2, we can obtain our desired results. □

COROLLARY 3.4. *Suppose that the conditions* (A) *and* (B) *hold. Let F be a pseudo-monotone C^r mapping from X into R^n. If there exists a point $x^0 \in X$ such that F is weakly proper at x^0, then the conclusion of Theorem* 3.1 *holds.*

Proof. Since F is weakly proper at x^0, for each sequence $\{x^k\} \subset X$ with the property $\|x^k\| \to \infty$ as $k \to \infty$, there exists some k such that $\|x^k\| > \|x^0\|$ and

$$(x^k - x^0)^T F(x^0) \geqslant 0$$

By the pseudo-monotonicity of F, we have

$$(x^k - x^0)^T F(x^k) \geqslant 0,$$

from which we can prove the conclusion of the corollary similarly with the proofs of Lemma 3.2, 3.3 and Theorem 3.1. □

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References

1. Pang, J.S. and Chan, D. (1982), Iterative method for variational and complementarity problem. *Mathematical Programming* 24, 284–313.
2. Harker, P.T. (1988), Accelerating the convergence of diagonalization and projection algorithms for finite-dimensional variational inequalities. *Mathematical Programming* 41, 29–59.
3. Kinderlehrer, D. and Stampacchia, G. (1980), *An Introduction to Variational Inequalities and Their Application*, Academic Press, New York.
4. Pang, J.S. (1994), Complementarity Problem. in: Horst, R. and Pardalos, P. (eds.), *Handbook of Global Optimization*, Kluwer Academic Publishers, Dordrecht, Netherlands.
5. Hartman, P. and Stampacchia, G. (1966), On some nonlinear elliptic differentiable functional equation. *Acta Mathematica* 115, 71–310.
6. Cottle, R.W. (1966), Nonlinear programs with positively bounded jacobins. *SIAM Journal on Applied Mathematics* 14, 147–158.
7. Eaves, B.C. (1971), On the basic theorem of complementarity problems. *Mathematical Programming* 1, 68–75.

8. Eaves, B.C. (1971), The linear complementarity problem. *Management Science* 17, 612–634.
9. Karamardian, S. (1972), The complementarity problems. *Mathematical Programming* 2, 107–129.
10. Lions, J.L. and Stampacchia, G. (1967), Variational inequalities. *Communications on Pure and Applied Mathematics* 20, 490–519.
11. Harker, P.T. and Pang, J.S. (1990), Finite-dimensional variational inequality and nonlinear complementarity problem: A survey of theory, algorithms and applications. *Mathematical Programming*, 48, 161–220.
12. Harker, P.T. and Pang, J.S. (1994), *Modelling and Computation of Equilibrium*: *A Variational Inequality Approach*. Academic Press, New York.
13. Allgower, E.L. and Georg K. (1990), *Numerical Continuation Methods*: *An Introduction*. Springer Verlag, Berlin, Germany.
14. Garcia, C.B. and Zangwill, W.I. (1981), *Pathways to Solutions*, *Fixed Points and Equilibria*. Prentice-Hall, New Jersey.
15. Bo, Yu. and Zhenghua, L. (1996), Homotopy method for a class of nonconvex Brouwer fixed point problems. *Applied Mathematics Computers* 74, 65–77.
16. Guochen, F., Zhenghua, L. and Bo, Yu. (1998), Existence of an interior pathway to a Karush-Kuhn-Tucker point of a nonlinear programming problem. *Nonlinear Analysis* 32, 761–768.
17. Lin, Z. and Li, Y. (1999), Homotopy method for solving variational inequalities. *Journal of Optimization Theory and Applications* 100(1), 207–218.
18. Naber, G.L. (1980), *Topological Method in Euclidean Space*. Cambridge University Press, London.
19. Zhao, Y.B., Han, J.Y. and Qi, H.D. (1999), Exceptional families and existence theorems for variation inequality problem. *Journal of Optimization Theory and Application* 101, 475–495.